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HYPERBOLIC METRIC, CURVATURE OF GEODESICS AND HYPERBOLIC DISCS IN HYPERBOLIC PLANE DOMAINS

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ABSTRACT
The purpose of this paper is to investigate the relations among some geometric quantities defined for every hyperbolic plane domain of any connectivity, each of which measures, in some sense, how much the domain deviates either from a disc, convex domain, or simply connected domain on one hand, or a punctured domain on the other hand.

Introduction

A domain D in the complex plane **C** is hyperbolic if there is an analytic covering map $f(z)$ of the unit disc Δ onto D. Every such covering projects onto D the hyperbolic (Poincaré) geometry in Δ and defines the hyperbolic metric $\rho_D(z)\,|\,dz\,|$, the hyperbolic geodesics and the hyperbolic discs in D, and also the following related quantities:

(1) $$\eta^*(z;D)=\rho_D(z)^{-1}\left|\frac{\partial}{\partial z}\log\rho_D(z)\right|,\qquad z\in D,$$

(2) $$\beta^*(z;D)=\rho_D(z)^{-1}\left|\frac{\partial^2}{\partial z^2}\rho_D(z)^{-1}\right|,\qquad z\in D,$$

(3) $$\tilde{\eta}(z;D)=\tfrac{1}{2}\rho_D(z)^{-1}\max\{k(z,\gamma):\gamma\in\Gamma(z;D)\},\qquad z\in D,$$

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and

$$\tilde{\beta}(z;D)=\tfrac{1}{2}\rho_D(z)^{-2}\max\left\{\frac{d}{ds}k(z,\gamma):\gamma\in\Gamma(z;D)\right\},\qquad z\in D. \tag{4}$$

where $\Gamma(z;D)$ denotes the family of all the hyperbolic geodesics γ in D such that $z\in\gamma$, $k(z,\gamma)$ is the Euclidean curvature of the curve γ at $z\in\gamma$, and d/ds denotes differentiation with respect to the arc-length parameter s on the curve γ. Also let

$$\eta(z;D)=\tfrac{1}{2}\left|(1-|\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)}-2\bar{\zeta}\right|,\qquad f(\zeta)=z\in D \tag{5}$$

and

$$\beta(z;D)=\tfrac{1}{2}(1-|\zeta|^2)^2|S_f(\zeta)|,\qquad f(\zeta)=z\in D, \tag{6}$$

where

$$S_f=\frac{f'''(\zeta)}{f'(\zeta)}-\frac{3}{2}\left(\frac{f''(\zeta)}{f'(\zeta)}\right)^2$$

is the Schwarzian derivative of some analytic covering $f(\zeta)=z$ of Δ onto D. Next define:

$$c(D)=\inf_{z\in D}\rho_D(z)d(z,\partial D), \tag{7}$$

where $d(z,\partial D)$ is the Euclidean distance of z from the boundary ∂D of D. Finally if $\Delta(z,r)$ denotes the hyperbolic disc in D centered at $z\in D$ and of a pseudo-hyperbolic radius r (i.e. $\Delta(z,r)$ is the image of the disc $\Delta_r=\{\zeta:|\zeta|<r\}$ under any analytic covering map $f(\zeta)$ of Δ onto D, such that $f(0)=z$), for some $r\in(0,1)$, we define the uniform radii of schlichtness and convexity of D, respectively, by

$$r_s(D)=\sup\{r>0:\Delta(z,r)\text{ is simply connected for every }z\in D\} \tag{8}$$

and

$$r_c(D)=\sup\{r>0:\Delta(z,r)\text{ is convex for every }z\in D\}. \tag{9}$$

In this paper we give a geometric interpretation for some results in [3], [6], [10], [14] and [15], showing that they imply relations among the nine quantities defined above, for any hyperbolic domain. In particular, using the fact that the uniform radius of schlichtness $r_s(D)$ is a conformal invariant, we conclude that there are either upper or lower bounds for each of the other quantities, depending merely on the conformal moduli of the domain, and in the case of doubly connected domains we establish explicit estimates for those bounds in terms of the conformal modulus of the domain. We conclude the paper with some internal differential-geometric conditions for a hyperbolic domain D to be either simply connected ($r_s(D) = 1$), q-quasidisc, convex ($r_c(D) = 1$) or k-convex (see [9] for definition), formulated either as differential inequalities for the hyperbolic metric $\rho_D(z)|dz|$, or in terms of the curvature of the hyperbolic geodesics in D.

1. Hyperbolic metric and curvature of geodesics

A domain D in $\mathbf{C}$ is hyperbolic if its universal covering space is the unit disc Δ. Any pair of analytic covering maps $f(z)$ and $\tilde{f}(z)$ of Δ onto D are related to each other by an identity of the form:

$$\tilde{f}(z) = f\left(\frac{\alpha z + \zeta}{1 + \alpha\bar{\zeta}z}\right), \qquad z \in \Delta, \tag{1.1}$$

for some $\alpha \in \partial\Delta$ and $\zeta \in \Delta$.

The hyperbolic metric (infinitesimal line element) $\rho_D(w)|dw|$ in D is defined as the projection of the hyperbolic (Poincaré) metric $(1 - |z|^2)^{-1}|dz|$ from Δ onto D, that is

$$\rho_D(w)|dw/dz| = (1 - |z|^2)^{-1} \qquad \text{for } w = f(z), \quad z \in \Delta, \tag{1.2}$$

where $f\colon \Delta \to D$ is some analytic covering map of Δ onto D.

Because of the relation (1.1) and the identity

$$|dw/dz| = (1 - |w|^2)/(1 - |z|^2) \qquad \text{for } w = (\alpha z + \zeta)/(1 + \alpha\bar{\zeta}z), \tag{1.3}$$

the definition of $\rho_D(w)$ by (1.2) is independent of the choice of the covering map $w = f(z)$. In particular we derive

$$\rho_D(w) = |\tilde{f}'(0)|^{-1} \qquad \text{for } w = f(\zeta) = \tilde{f}(0) \in D. \tag{1.2'}$$

Now, let w be any point in D, and $f\colon \Delta \to D$ a covering map of Δ onto D. Let $\zeta \in \Delta$ be such that $f(\zeta) = w$. A *hyperbolic geodesic* γ in D passing through w is the image of the straight line segment $\tilde{\gamma} = \{z = \alpha t : t \in (-1, 1)\}$, for some $\alpha \in \partial\Delta$, under the map $\tilde{f}(z) = f((z + \zeta)/(1 + \bar{\zeta}z))$. Notice that both α and $-\alpha$ determine the same geodesic γ in D, just inversely oriented.

The (Euclidean, signed) curvature $k(\omega,\gamma)$ of a C^2-curve γ at a point $\omega\in\gamma$ is the rate of change in the direction of the tangent vector (positively oriented) to γ at ω, with respect to the Euclidean arc-length parameter s on γ, i.e.

$$(1.4)\quad k(\omega,\gamma)=|w'(t)|^{-1}\frac{d}{dt}(\arg w'(t))=|w'(t)|^{-1}\operatorname{Im}(w''(t)/w'(t)),\quad t=t_0,$$

where $w=w(t)$, $t\in(a,b)$, is a C^2-parametrization of γ with $\omega=w(t_0)$ and $w'(t_0)\neq 0$.

Next, in order to obtain expressions for the curvature and its arc-length derivative along a hyperbolic geodesic at a given point, we notice, first, that the two operators

$$\Lambda_1(f,z)=\frac{f''(z)}{f'(z)}-\frac{2\bar{z}}{1-|z|^2}$$

and

$$\Lambda_2(f,z)=\frac{f'''(z)}{f'(z)}-\frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

are covariant derivatives of the first and the second orders, respectively, in the sense that they satisfy the relation

$$(1.5)\qquad \Lambda_n(f\circ g,z)=\Lambda_n(f,g(z))g'(z)^n,\qquad n=1,2,$$

for every $g(z)=(\alpha z+\zeta)/(1+\alpha\bar{\zeta}z)$, $\alpha\in\partial\Delta$, $\zeta\in\Delta$ (see [6] for $n=1$ and [8] p. 52 for $n=2$).

Proposition 1. *Let γ be a hyperbolic geodesic given by:*

$$\gamma=\{w=w(t)=\tilde{f}(t)=f((\alpha t+\zeta)/(1+\alpha\bar{\zeta}t)):-1<t<1\},$$

for some $\alpha\in\partial\Delta$ and $\zeta\in\Delta$. Then

$$(1.6)\qquad \rho_D(w)^{-1}k(w,\gamma)=(1-|\zeta|^2)\operatorname{Im}(\alpha\Lambda_1(f,\zeta)),$$

and

$$(1.7)\quad \rho_D(w)^{-2}\frac{d}{ds}k(w,\gamma)=(1-|\zeta|^2)^2\operatorname{Im}(\alpha^2\Lambda_2(f,\zeta))\quad \text{(cf. [1] p. 21)}$$

where $w = f(\zeta)$.

PROOF. From the parametrization of the geodesic γ we derive:

$$w(0) = \tilde{f}(0) = f(\zeta) = w,$$

$$w'(0) = \tilde{f}'(0) = \alpha(1 - |\zeta|^2)f'(\zeta) \Rightarrow |w'(0)| = \rho_D(w)^{-1},$$

$$w''(0)/w'(0) = \tilde{f}''(0)/\tilde{f}'(0) = \Lambda_1(\tilde{f}, 0) = \alpha(1 - |\zeta|^2)\Lambda_1(f, \zeta)$$

and

$$\Lambda_2(w, 0) = \Lambda_2(\tilde{f}, 0) = \alpha^2(1 - |\zeta|^2)^2\Lambda_2(f, \zeta).$$

From formula (1.4) we obtain:

$$\rho_D(w)^{-1}k(w, \gamma) = \mathrm{Im}(w''(0)/w'(0)) = \mathrm{Im}(\alpha(1 - |\zeta|^2)\Lambda_1(f, \zeta)).$$

Differentiation of (1.4) with respect to the arc-length s yields:

$$\begin{aligned} |w'(t)|^2 \frac{d}{ds} k(w(t), \gamma) &= |w'(t)|^2 \mathrm{Im}\left(\frac{d}{ds}(|w'(t)|^{-1} w''(t)/w'(t))\right) \\ &= |w'(t)|\,\mathrm{Im}\left(\frac{d}{dt}(w''(t)w'(t)^{-3/2}\overline{w'(t)}^{\,-1/2})\right) \\ &= \mathrm{Im}\left(\frac{w'''(t)}{w'(t)} - \frac{3}{2}\left(\frac{w''(t)}{w'(t)}\right)^2\right) \\ &= \mathrm{Im}(\Lambda_2(w, t)). \end{aligned}$$

Hence we get at $t = 0$:

$$\begin{aligned} \rho_D(w)^{-2}\frac{d}{ds}(k(w, \gamma) &= |w'(0)|^2 \frac{d}{ds} k(w, \gamma) \\ &= \mathrm{Im}(\Lambda_2(w, 0)) \\ &= \mathrm{Im}(\alpha^2(1 - |\zeta|^2)^2\Lambda_2(f, \zeta)). \end{aligned}$$

q.e.d.

COROLLARY 1. *For each point z in a hyperbolic domain D there are two geodesics γ_1 and γ_2 passing through z, such that*:

(1.8) $$k(z, \gamma_1) = \max\{|k(z, \gamma)| : \gamma \in \Gamma(z; D)\} = |f'(\zeta)^{-1}\Lambda_1(f, \zeta)|,$$

and

(1.9) $$\frac{d}{ds}k(z,\gamma_2)=\max\left\{\left|\frac{dk(z,\gamma)}{ds}\right| : \gamma\in\Gamma(z;D)\right\}=|f'(\zeta)^{-2}\Lambda_2(f,\zeta)|,$$

where $f(\zeta)=z$.

PROOF. Formulas (1.6) and (1.7) easily yield:

(1.6′) $$k(z,\gamma)\leqq\rho_D(z)(1-|\zeta|^2)|\Lambda_1(f,\zeta)|,\qquad f(\zeta)=z$$

and

(1.7′) $$\frac{d}{ds}k(z,\gamma)\leqq\rho_D(z)^2(1-|\zeta|^2)^2|\Lambda_2(f,\zeta)|,\qquad f(\zeta)=z.$$

with equalities only on the corresponding geodesics:

$$\gamma_j=\{w=f((\alpha_j t+\zeta)/(1+\alpha_j\bar{\zeta}t)),\ -1<t<1\},\qquad j=1,2,$$

where the α_j, $j=1,2$, are determined by:

$$\operatorname{Im}(\alpha_j^j\Lambda_j(f,\zeta))=|\Lambda_j(f,\zeta)|,\qquad j=1,2,$$

i.e.

$$\arg\alpha_j=\frac{\pi}{2j}-\frac{1}{j}\arg\Lambda_j(f,\zeta),\qquad j=1,2.$$

Finally, by formula (1.2) we have:

$$\rho_D(z)(1-|\zeta|^2)=|f'(\zeta)|^{-1},\qquad f(\zeta)=z,$$

and thus inequalities (1.6′) and (1.7′) readily imply (1.8) and (1.9). q.e.d.

THEOREM 1. *Let D be a hyperbolic domain in* **C**. *Then*:

(1.10) $$\eta(z;D)=\eta^*(z;D)=\tilde{\eta}(z;D),\qquad z\in D$$

and

(1.11) $$\beta(z;D)=\beta^*(z;D)=\tilde{\beta}(z;D),\qquad z\in D.$$

PROOF. From the proof of Corollary 1 we already have $\tilde{\eta}(z;D)=\eta(z;D)$ and $\tilde{\beta}(z;D)=\beta(z;D)$ for all $z\in D$. Next, if we differentiate the identity:

$$\log\rho_D(z)=-\log|f'(\zeta)|-\log(1-|\zeta|^2),\qquad f(\zeta)=z.$$

with respect to ζ, we obtain:

$$f'(\zeta)\frac{\partial}{\partial z}\log\rho_D(z) = -\frac{1}{2}\left[\frac{f''(\zeta)}{f'(\zeta)} - \frac{2\bar{\zeta}}{1-|\zeta|^2}\right] = -\frac{1}{2}\Lambda_1(f,\zeta),$$

which proves (1.10). Next we apply the identity:

$$\Lambda_2(f,\zeta)f'(\zeta)^{-2}$$
$$= \frac{\partial}{\partial\zeta}[\Lambda_1(f,\zeta)f'(\zeta)^{-1}]f'(\zeta)^{-1} + \tfrac{1}{2}[\Lambda_1(f,\zeta)f'(\zeta)^{-1}]^2$$

and conclude

$$\Lambda_2(f,\zeta)f'(\zeta)^{-2} = -2\frac{\partial^2}{\partial z^2}\log\rho_D(z) + 2\left[\frac{\partial}{\partial z}\log\rho_D(z)\right]^2$$
$$= 2\rho_D(z)\frac{\partial^2}{\partial z^2}[\rho_D(z)^{-1}]$$

and identities (1.11) follows. q.e.d.

Theorems 3, 4 and 5 in [13] imply:

THEOREM 2. *Let D be a hyperbolic domain with $c(D) > 0$. Then*

$$(1.12)\qquad c(D)^{-1} \leqq 2\eta(D) = \sup_{|z|<1}\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right| \leqq 2c(D)^{-1}$$

hold for every analytic covering $f(z)$ of Δ onto D.
Moreover, if D is simply connected, then:

$$(1.13)\quad \eta(\zeta;D) = \tfrac{1}{2}\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right| \leqq \min\{2, \tfrac{3}{2}c(D)^{-1}\},\ \zeta = f(z)\in D.$$

PROOF. From Theorem 5 in [13] it follows that:

$$(1.14)\qquad \eta^*(D) = \sup_{z\in D}\eta^*(z;D) \geqq \tfrac{1}{2}c(D)^{-1}$$

On the other hand, inequalities (20), (14) and (15) in [13] may be rewritten, in our notation, as follows:

$$(1.15)\qquad \eta^*(z;D) \leqq \rho_D(z)^{-1}d(z,\partial D)^{-1}, \qquad z\in D,$$

for every hyperbolic domain D, and for a simply connected domain:

$$(1.13^*)\qquad \eta^*(z;D) \leqq \min\{2, \tfrac{3}{2}\rho_D(z)^{-1}d(z,D)^{-1}\}, \qquad z\in D.$$

Thus, in view of (1.10), inequalities (1.14), (1.15) and (1.13*) imply the LHS, the RHS of (1.12) and (1.13), respectively. q.e.d.

Folgerung 2.3 in [14] and Theorem 3 in [6] yield the following relations between the η's and the β's:

THEOREM 3. *Let D be a hyperbolic domain in* $\mathbf{C}$.

(i) *If $\eta(D)=\sup_{z\in D}\eta(z;D)$ is given, then*:

$$\beta(D)=\sup_{z\in D}\beta(z;D)\leqq\beta(\eta(D))\leqq\eta(D)^2+1, \tag{1.16}$$

and

$$\sup_{z\in D}\{\beta(z;D)+\eta(z;D)^2\}\leqq p(\eta(D)), \tag{1.17}$$

where

$$\beta(\eta)^2=\begin{cases}[(27\eta^4-18\eta^2-1)+(\eta^2-1)^{1/2}(9\eta^2-1)^{3/2}]/8\eta^2, & 1\leqq\eta\leqq\sqrt{1+\sqrt{2}},\\ (\eta^2+1)^2, & \eta\geqq\sqrt{1+\sqrt{2}},\end{cases}$$

and

$$p(\eta)=\eta^2+(\eta+\tfrac{1}{2}\eta^{-1})\sqrt{\eta^2-1}.$$

(ii) *If $\beta(D)$ is given, then*:

$$\eta(D)\leqq\sqrt{\beta(D)+1}. \tag{1.18}$$

By Theorem 1, Theorem 3 may be formulated now completely in terms of the hyperbolic metric in D:

THEOREM 3*. *If the differential inequality*

$$\rho_D(z)^{-1}\left|\frac{\partial}{\partial z}\log\rho_D(z)\right|\leqq\eta$$

holds in D for some $\eta\geqq 1$, then:

$$\rho_D(z)^{-1}\left|\frac{\partial^2}{\partial z^2}\rho_D(z)^{-1}\right|\leqq\beta(\eta)\leqq\eta^2+1,\qquad z\in D, \tag{1.16*}$$

and

$$\rho_D(z)^{-2}\left|\frac{\partial^2}{\partial z^2}\log\rho_D(z)\right|\leqq p(\eta),\qquad z\in D. \tag{1.17*}$$

Conversely, if $\rho_D(z)$ *satisfies the differential inequality*

$$\rho_D(z)^{-1}\left|\frac{\partial^2}{\partial z^2}\rho_D(z)^{-1}\right| \leqq \beta, \qquad z \in D,$$

then:

$$\rho_D(z)^{-1}\left|\frac{\partial}{\partial z}\log\rho_D(z)\right| \leqq \sqrt{\beta+1}, \qquad z \in D. \tag{1.18*}$$

2. Uniform radii of schlichtness and convexity

By definition (see introduction) a hyperbolic domain D has a uniform radius of schlichtness $r_s(D)$ (of convexity $r_c(D)$), if there is an analytic covering $f(z)$ of Δ onto D which is univalent (convex, respectively) on every hyperbolic disc in Δ of a pseudo-hyperbolic radius r (i.e. in every disc $|(z-\zeta)/(1-\bar{\zeta}z)| < r$, for any $\zeta \in \Delta$), for every radius $r < r_s(D)$ ($r < r_c(D)$, respectively), but not for larger radii. In view of identity (1.1), the definition of both $r_s(D)$ and $r_c(D)$ is independent of the choice of the covering map $f(z)$, and therefore they represent intrinsic, geometric properties of D. By using a generalization of Nehari's [11] univalence criteria, due to Schwarz [15], Beesack–Schwarz [3] and Minda [10], and applying identities (1.11), we derive estimates for $r_s(D)$ in terms of the hyperbolic metric in D, namely:

THEOREM 4. *Let D be a hyperbolic domain with $\beta^*(D) \geqq 1$. Then*

$$\tanh(\pi/[2\sqrt{\beta^*(D)-1}]) \leqq r_s(D) \leqq \min(1, \sqrt{3/\beta^*(D)}), \tag{2.1}$$

where

$$\beta^*(D) = \sup_{z\in D}\rho_D(z)^{-1}\left|\frac{\partial^2}{\partial z^2}\rho_D(z)^{-1}\right|.$$

PROOF. By Theorem 1 the condition $\beta^*(D) \geqq 1$ is equivalent to

$$\sup_{|z|<1}(1-|z|^2)^2|S_f(z) = 2\beta(D) \geqq 2$$

for every analytic covering map $f(z)$ of Δ onto D. Hence, Theorem 1 in [3] (or Theorem 3 in [10]) shows that $f(z)$ is univalent in all hyperbolic discs in Δ of a hyperbolic radius $\pi/[2\sqrt{\beta(D)-1}]$, which implies the LHS of (2.1). On the other hand, by Theorem 4 in [15] (or Theorem 4 in [10]), if $f(z)$ is univalent in every disc in Δ of a hyperbolic radius $\frac{1}{2}\log[(1+r)/(1-r)]$, then

$$2\beta^*(D) = 2\beta(D) \leqq 6/r^2,$$

and the RHS of (2.1) follows at once. q.e.d.

THEOREM 5. *Let D be a hyperbolic domain in* **C**. *Then $\eta^*(D) \geqq 1$ and*

$$\begin{aligned}\eta^*(D) - \sqrt{\eta^*(D)^2 - 1} \leqq r_c(D) \leqq \sigma^*(D)^{-1/2} \\ \leqq \min\{\beta^*(D)^{-1/2}, \eta^*(D)^{-1}\},\end{aligned} \tag{2.2}$$

where

$$\sigma^*(D) = \sup_{|z|<1} [\beta^*(z; D) + \eta^*(z; D)^2].$$

PROOF. First, if we assume that $\eta^*(D) = \eta < 1$, then by Theorem 1

$$\left| z\frac{f''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2} \right| \leqq \frac{2\eta |z|}{1-|z|^2}$$

holds for every analytic covering map $f(z)$ of Δ onto D and all $z \in \Delta$, and hence $\mathrm{Re}[zf''(z)/f'(z)] \geqq 2|z|(|z| - \eta)/(1-|z|^2)$ for all $z \in \Delta$. Therefore, if $\eta < |z| < 1$, then $\mathrm{Re}[zf''(z)/f'(z)] > 0$, contradicting the mean value property of harmonic functions, since at $z = 0$ we have $\mathrm{Re}[zf''(z)/f'(z)] = 0$. Hence $\eta^*(D) \geqq 1$ for every hyperbolic domain D. By Satz 2.5 in [14] it follows that $f(z)$ is convex in every disc in Δ with the pseudo-hyperbolic radius $r = \eta - \sqrt{\eta^2 - 1}$, where $\eta = \eta(D)$, and therefore the LHS of (2.2) follows by identity (1.10). Conversely, if $f(z)$ is convex in every disc in Δ of a pseudo-hyperbolic radius r, then by Theorem 2 in [6], the inequality

$$(1-|z|^2)^2\{|\Lambda_2(f, z)| + \tfrac{1}{2}|\Lambda_1(f, z)|^2\} \leqq 2/r^2, \qquad |z| < 1$$

holds, and by identity (1.10) it is identical with the RHS of (2.2). q.e.d.

Theorems 3, 4 and 5 yield the following relations between the radii of schlichtness and convexity:

COROLLARY 2. *Let D be a hyperbolic domain with uniform radii of schlichtness $R = r_s(D)$ and convexity $r = r_c(D)$. Then*

$$R \geqq \tanh(\pi r/[2\sqrt{1-r^2}]) \tag{2.3}$$

and

$$r \geqq (\sqrt{3} + \sqrt{3 + R^2})^{-1} R \geqq (2 - \sqrt{3})R \tag{2.4}$$

and therefore $R > 0$ if and only if $r > 0$.

PROOF. Inequality (2.3) is a composition of the LHS of (2.1) and the RHS of (2.2), and (2.4) follows from the LHS of (2.2), (1.18) and the RHS of (2.1). q.e.d.

From Theorems 3, 4 and 5 we can also derive upper bounds for the η's and the β's of D in terms of $r_s(D)$ and $r_c(D)$:

COROLLARY 3. *Let D be a hyperbolic domain with $R = r_s(D)$ and $r = r_c(D)$. Then the following inequalities hold at all points $z \in D$:*

$$\tfrac{1}{2}\max\{k(z,\gamma):\gamma\in\Gamma(z,D)\} = \left|\frac{\partial}{\partial z}\log\rho_D(z)\right| \leqq \min\{r^{-1},\sqrt{1+3R^{-2}}\}\rho_D(z), \tag{2.5}$$

$$\tfrac{1}{2}\max\left\{\frac{d}{ds}k(z,\gamma):\gamma\in\Gamma(z;D)\right\} = \rho_D(z)\left|\frac{\partial^2}{\partial z^2}\rho_D(z)^{-1}\right| \leqq \min\{r^{-2},3R^{-2}\}\rho_D(z)^2 \tag{2.6}$$

and

$$\rho_D(z)^{-2}\left|\frac{\partial^2}{\partial z^2}\log\rho_D(z)\right| \leqq \sigma^*(D) \leqq 1/r^2. \tag{2.7}$$

Inequality (2.5) with Theorem 5 in [13] imply an improvement on the left-hand inequality of Theorem 5 in [10], namely:

COROLLARY 4. *If $R = r_s(D)$ and $r = r_c(D)$ for a hyperbolic plane domain D, then:*

$$c(D) = \inf_{z\in D}\rho_D(z)d(z,\partial D) \geqq \tfrac{1}{2}\max\{r, R/\sqrt{3+R^2}\}. \tag{2.8}$$

In the opposite direction, inequalities (2.1), (2.2), (1.12) and (1.16) yield lower bounds for the uniform radii of schlichtness and convexity of a hyperbolic domain D, in terms of $c(D)$:

COROLLARY 5. *Let D be a hyperbolic domain with $c(D) > 0$. Then*

$$r_s(D) \geqq \tanh(\tfrac{1}{2}\pi c(D)) > 0 \tag{2.9}$$

and

(2.10) $$r_c(D) \geqq c(D)^{-1} - \sqrt{c(D)^{-2} - 1} > \tfrac{1}{2}c(D) > 0.$$

REMARK. Among all the domain constants discussed above, $r_s(D)$ is of special importance because of its invariance under conformal homeomorphisms, which follows from the facts that hyperbolic discs are mapped onto hyperbolic discs with the same (pseudo-)hyperbolic radii, by conformal homeomorphisms, and their connectivity is preserved. Therefore $r_s(D)$ depends only on the conformal moduli of D. Thus, once $r_s(D)$ is expressed in terms of the conformal moduli, the results above provide bounds for all the other quantities for all the conformally equivalent domains, that is for all the domains with the same conformal moduli.

LEMMA 1. *Let D be a doubly connected domain with a conformal modulus $m > 0$, that is D is conformally equivalent to the annulus $A_m = \{z\mathbf{C} : e^{-m\pi} < |z| < e^{m\pi}\}$. Then*

(2.11) $$r_s(D) = \tanh(\pi/4m).$$

PROOF. By the remark above, it is enough to show that (2.11) holds for $D = A_m$, since $r_s(D) = r_s(A_m)$.

Let $f(z) = g \circ h(z)$, where $\zeta = h(z) = 2mi \log[(1 - z)/(1 + z)]$ and $w = g(\zeta) = e^{\zeta}$. Then $\zeta = h(z)$ maps Δ conformally onto the infinite strip domain:

$$\mathbf{T}_m = \{\zeta : |\mathrm{Re}\, \zeta| < m\pi\}$$

and $w = g(\zeta)$ is an analytic covering map of $\mathbf{T}_m$ onto A_m. Hence $w = f(z)$ is an analytic covering map of Δ onto A_m. Because of the rotational symmetry of A_m, it is enough to consider hyperbolic discs $\Delta(e^{m\varphi}, r)$, centered at points $e^{m\varphi}$, for $\varphi \in (-\pi, \pi)$, and of a pseudo-hyperbolic radius $r > 0$. The hyperbolic disc $\Delta(e^{m\varphi}, r)$ is the image of the disc

$$\tilde{\Delta}(i\lambda, r) = \left\{z : \left|\frac{z - i\lambda}{1 + i\lambda z}\right| < z\right\}, \qquad \lambda = \tan(\tfrac{1}{4}\varphi),$$

under $w = f(z)$. But one can show that the conformal mapping $\zeta = h(z)$ maps $\tilde{\Delta}(i\lambda, r)$ into the strip $\{\zeta : |\mathrm{Im}\, \zeta| < 2m \log[(1 + r)/(1 - r)]\}$, where the disc $\tilde{\Delta}(0, r)$, in particular, cannot be mapped into a narrower strip. Hence, in order that $\Delta(e^{m\varphi}, r)$ would be simply connected, for each $\varphi \in (-\pi, \pi)$, r should satisfy the inequality:

$$2m \log \frac{1 + r}{1 - r} \leqq \pi,$$

that is $r \leqq \tanh(\pi/4m)$. On the other hand, the two points $\pm\tanh(\pi/4m)$, which are mapped by $f(z)$ into the same point $w = -1$, show that $r_s(D) = \tanh(\pi/4m)$. q.e.d.

Thus, Theorems 3 and 4 readily yield:

THEOREM 6. *Let D be a doubly connected domain with a conformal modulus $m > 0$. If $f(z)$ is an analytic covering map of Δ onto D, then*:

$$1 + 4m^2 \leqq \tfrac{1}{2} \sup_{|z|<1} (1 - |z|^2)^2 |S_f(z)| \leqq 3\coth^2(\pi/4m) \tag{2.12}$$

and

$$\max(1, 2m) \leqq \tfrac{1}{2} \sup_{|z|<1} \left|(1 - |z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right| \leqq \sqrt{1 + 3\coth^2(\pi/4m)}. \tag{2.13}$$

We conclude this section with the following estimates for the two domain constants $r_c(D)$ and $c(D)$ for doubly-connected domains with a given conformal modulus:

THEOREM 7. *If D is a doubly-connected domain of coformal modulus $m > 0$, then*:

$$\tfrac{1}{2}[1 + 3\coth^2(\pi/4m)]^{-1/2} \leqq \inf_{z \in D} \rho_D(z) d(z, \partial D) \leqq \tfrac{1}{2}\min(1, 1/m) \tag{2.14}$$

and

$$(2 - \sqrt{3})\tanh(\pi/4m) < r_c(D) \leqq (1 + 4m^2)^{-1/2}. \tag{2.15}$$

PROOF. From Theorem 2 in [9] it follows that $c(D) < \frac{1}{2}$ for every non-convex hyperbolic domain D. Furthermore, inequality (2.9) and identity (2.11) imply: $c(D) \leqq 1/2m$. The LHS of (2.14) follows from (2.8) and (2.11). Corollary 2 with Lemma 1 easily yield inequalities (2.15). q.e.d.

3. Simply connected and convex domains

In this section we apply Theorem 1 to various already known results relating geometric and topological properties of the image domain $D = f(\Delta)$ of an analytic function $f(z)$ in the unit disc Δ, such as simple-connectivity, quasi-circularity and convexity, to the functionals:

$$\sup_{|z|<1} (1-|z|^2)^j |\Lambda_j(f,z)|, \qquad j=1,2,$$

and rewrite them either in terms of the curvature of the geodesics or as differential inequalities on the hyperbolic metric in D. First, Pommerenke's [14], Osgood's [13] and the author's improvement of Nehari's univalence condition [11] imply:

THEOREM 8. *Let D be a simply connected hyperbolic domain in* **C**. *Then the following inequalities hold in D*:

$$\rho_D(z)^{-1}\left|\frac{\partial^2}{\partial z^2}\rho_D(z)^{-1}\right| \leqq 3\delta(z;D) \leqq 3, \qquad z\in D, \tag{3.1}$$

$$\rho_D(z)^{-1}\left|\frac{\partial}{\partial z}\log\rho_D(z)\right| \leqq \sqrt{1+3\delta(D)} \leqq 2, \qquad z\in D, \tag{3.2}$$

$$\rho_D(z)d(z,\partial D) \geqq \tfrac{1}{2}(1+3\delta(D))^{-1/2} \geqq \tfrac{1}{4}, \qquad z\in D, \tag{3.3}$$

and

$$r_c(D) \geqq \sqrt{1+3\delta(D)} - \sqrt{3\delta(D)} \geqq 2-\sqrt{3}, \tag{3.4}$$

where (see [5])

$$\delta(z;D) = \{1-\pi\rho_D(z)^{-2}\Gamma_D(z)\}^{1/2}, \quad \Gamma_D(z) = \frac{1}{\pi^2}\iint_{\mathbf{C}\setminus D} |\zeta - z|^{-4} d\xi d\eta,$$

$$\zeta = \xi + i\eta$$

and $\delta(D) = \sup_{z\in D}\delta(z;D)$.

PROOF. If D is a simply connected domain, there is a univalent conformal mapping $f(\zeta)$ of Δ onto D, and by Theorem 2 in [5]:

$$(1-|\zeta|^2)^2|\Lambda_2(f,\zeta)| \leqq 6\delta(f(\zeta);D), \qquad \zeta\in\Delta. \tag{3.1'}$$

But by identity (1.10), inequalities (3.1) and (3.1′) are identical. Inequality (3.2) follows from (3.1) by the second part of Theorem 3*. (3.2) implies (3.3) by Theorem 5 in [13], and it implies (3.4) by the LHS of (2.2). q.e.d.

A domain D in $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is a *q-quasidisc*, for some $q\in[0,1)$, if it is an image of Δ under a q-quasiconformal automorphism of $\hat{\mathbf{C}}$, that is conformal in Δ. By Corollary 1 in [5], $\delta(D) \leqq q$ for every q-quasidisc D. Thus one can easily derive from Theorem 8 estimates for $\beta^*(D)$, $\eta^*(D)$, $c(D)$ and $r_c(D)$, for every q-quasidisc D, $0 \leqq q < 1$.

Conversely, by Gehring–Pommerenke's [4] and Ahlfors–Weill's [2] improvements of Nehari's criterion for univalence we readily get:

THEOREM 9. *Let D be a hyperbolic domain in* **C**. *If either one of the following equivalent conditions,*

$$\beta^*(z; D) = \rho_D(z)^{-1} \left| \frac{\partial^2}{\partial z^2} \rho_D(z)^{-1} \right| \leqq 1 \tag{3.5}$$

or

$$\tilde{\beta}(z; D) = \tfrac{1}{2}\rho_D(z)^{-2} \max \left\{ \frac{d}{ds} k(z, \gamma) : \gamma \in \Gamma(z; D) \right\} \leqq 1, \tag{3.5'}$$

holds for all $z \in D$, *then D is either a Jordan domain or the image under some Möbius transformation of the parallel strip domain* $\mathbf{T} = \{w : |\mathrm{Im}\, w| < \pi/2\}$, *in which case, and only in that case, an equality holds at some point in D.*

If the upper bound in (3.5) *and* (3.5′) *is replaced by* $q \in [0, 1)$, *then D is a q-quasidisc.*

Following Mejia–Minda [9] we call a domain D *k-convex* for some $k > 0$, if diam $D \leqq 2/k$ and D contains the closed lens-shaped region bounded between the two shorter circular arcs of radius $1/k$ connecting any pair of points a, b in D. Also, an analytic function $f(z)$ in Δ is k-convex if $f'(z) \neq 0$ in Δ and $D = f(\Delta)$ is k-convex.

Thoerem 8 and Corollary 1 in ([9], section 8) are refinements of the classical convexity criteria:

$$(1 - |z|^2)|\Lambda_1(f, z)| \leqq 2, \qquad z \in \Delta, \tag{3.6}$$

and

$$1 + \mathrm{Re}\, z \frac{f''(z)}{f'(z)} > 0, \qquad z \in \Delta. \tag{3.7}$$

The following result is an improvement on those refined criteria for k-convexity.

THEOREM 10. *An analytic function* $f(z)$ *is k-convex in* Δ *if and only if it satisfies either one of the following two inequalities*:

$$(3.8)\qquad (1-|z|^2)|\Lambda_1(f,z)| \leqq 2(1-kd(f(z),\partial D)), \qquad z\in\Delta,$$

and

$$(3.9)\qquad 1+\operatorname{Re} z\frac{f''(z)}{f'(z)} > \frac{2k|z|}{1-|z|^2}d(f(z),\partial D)), \qquad z\in\Delta,$$

and if one of these two inequalities holds (and therefore both of them), then:

$$(3.10)\qquad \begin{aligned} 1+\operatorname{Re} z\frac{f''(z)}{f'(z)} &\geqq \frac{kd(f(z),\partial D)(2-kd(f(z),\partial D))}{1-|z|^2} \\ &\geqq k|f'(z)|, \quad z\in\Delta. \end{aligned}$$

Proof. By Corollary 1 in ([9], section 7), the curvature of all the hyperbolic geodesics γ in a k-convex domain D is bounded by:

$$(3.11)\qquad |k(w,\gamma)| \leqq \frac{2(1-kd(w,\partial D))}{d(w,\partial D)(2-kd(w,\partial D))}, \qquad w\in\gamma\subset D.$$

On the other hand, by Theorem 1 in [9], we also have:

$$(3.12)\qquad \rho_D(w)^{-1} \leqq d(w,\partial D)(2-kd(w,\partial D)), \qquad w\in D.$$

Hence, inequality (3.8) follows easily from (3.11) and (3.12), using (1.10).

Next, inequality (3.8) shows that for every $z\in\Delta$, the value of $w = 1+zf''(z)/f'(z)$ lies in the closed disc

$$\left| w - \frac{1+|z|^2}{1-|z|^2}\right| \leqq \frac{2|z|}{1-|z|^2}(1-kd(f(z),\partial D))$$

and hence

$$\operatorname{Re} w \geqq \frac{1+|z|^2-2|z|}{1-|z|^2} + \frac{2k|z|}{1-|z|^2}d(f(z),\partial D),$$

which yields (3.9).

Now, inequality (3.9) implies the convexity criterion (3.7), i.e. $D=f(\Delta)$ is convex. Hence, by Theorem 2 in [9] (or (3.12) for $k=0$):

$$d(f(z),\partial D) \geqq \tfrac{1}{2}\rho_D(f(z))^{-1} = \tfrac{1}{2}(1-|z|^2)|f'(z)|, \qquad z\in\Delta$$

and hence inequality (3.9) implies:

$$1 + \operatorname{Re} z \frac{f''(z)}{f'(z)} > k\,|zf'(z)|, \qquad z \in \Delta,$$

which is a sufficient condition for k-convexity of $f(z)$, by Corollary 1 ([9], section 8).

Finally, if we square inequality (3.8) we obtain:

$$0 \leqq (1 - |z|^2) \left|\frac{f''(z)}{f'(z)}\right|^2$$
$$\leqq 4\left\{1 + \operatorname{Re} z \frac{f''(z)}{f'(z)} - \frac{kd(f(z), \partial D)(2 - kd(f(z), \partial D))}{1 - |z|^2}\right\},$$

from which the left-hand inequality in (3.10) follows. The right-hand inequality of (3.10) follows by (3.12) and (1.2). q.e.d.

By ([9], Corollary 2 in Section 7 and Theorem 1), the hyperbolic metric in a k-convex domain satisfies the inequality:

$$\rho_D(z)^{-1} \left|\frac{\partial}{\partial z} \log \rho_D(z)\right| \leqq 1 - kd(z, \partial D), \qquad z \in D. \tag{3.13}$$

Theorem 10 and Theorem 1 show that (3.13) is not only a necessary but also a sufficient condition for k-convexity of D.

Corollary 6. *A hyperbolic domain D is k-convex, for some $k > 0$, if and only if the hyperbolic metric $\rho_D(z)$ satisfies* (3.13).

In particular, in the case $k = 0$ we conclude:

Corollary 7. *A hyperbolic domain D in $\mathbf{C}$ is convex if and only if its hyperbolic metric $\rho_D(z)$ satisfies the differential inequality*:

$$\left|\frac{\partial}{\partial z} \log \rho_D(z)\right| \leqq \rho_D(z), \qquad z \in D, \tag{3.14}$$

and if it holds then we also have (*by Theorem* 3):

$$\left|\frac{\partial^2}{\partial z^2} \log \rho_D(z)\right| \leqq \rho_D(z) \left|\frac{\partial^2}{\partial z^2} \rho_D(z)^{-1}\right| + \left|\frac{\partial}{\partial z} \log \rho_D(z)\right|^2 \leqq \rho_D(z)^2, \quad z \in D. \tag{3.15}$$

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